Force System Resultants

4

CHAPTER OBJECTIVES

- To discuss the concept of the moment of a force and show how to calculate it in two and three dimensions.
- To provide a method for finding the moment of a force about a specified axis.
- To define the moment of a couple.
- To present methods for determining the resultants of nonconcurrent force systems.
- To indicate how to reduce a simple distributed loading to a resultant force having a specified location.

4.1 Moment of a Force—Scalar Formulation

When a force is applied to a body it will produce a tendency for the body to rotate about a point that is not on the line of action of the force. This tendency to rotate is sometimes called a *torque*, but most often it is called the moment of a force or simply the *moment*. For example, consider a wrench used to unscrew the bolt in Fig. 4–1*a*. If a force is applied to the handle of the wrench it will tend to turn the bolt about point O (or the z axis). The magnitude of the moment is directly proportional to the magnitude of $\mathbf{F}$ and the perpendicular distance or *moment arm* d. The larger the force or the longer the moment arm, the greater the moment or turning effect. Note that if the force $\mathbf{F}$ is applied at an angle $\theta \neq 90°$, Fig. 4–1*b*, then it will be more difficult to turn the bolt since the moment arm $d' = d\sin\theta$ will be smaller than d. If $\mathbf{F}$ is applied along the wrench, Fig. 4–1*c*, its moment arm will be zero since the line of action of $\mathbf{F}$ will intersect point O (the z axis). As a result, the moment of $\mathbf{F}$ about O is also zero and no turning can occur.

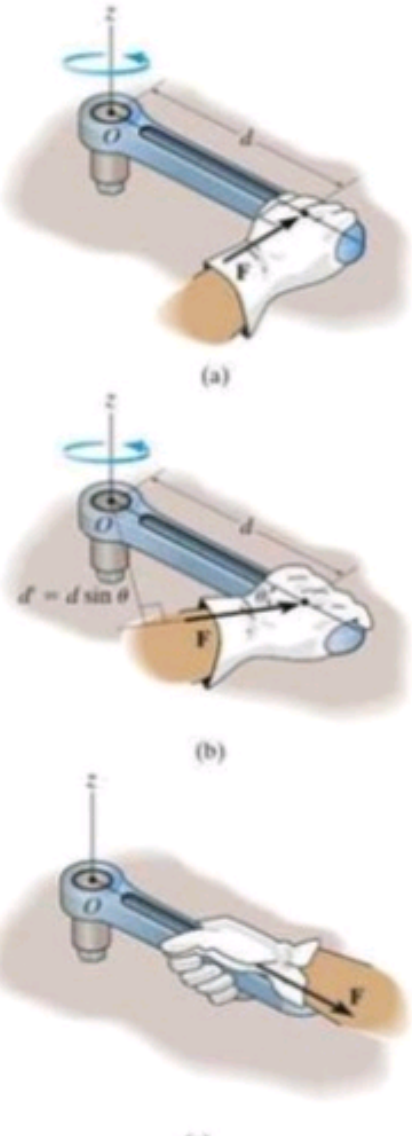

Fig. 4–1

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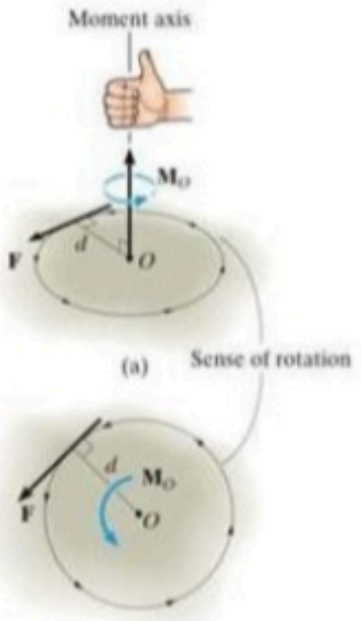

Fig. 4–2

We can generalize the above discussion and consider the force **F** and point O which lie in the shaded plane as shown in Fig. 4–2*a*. The moment $\mathbf{M}_O$ about point O, or about an axis passing through O and perpendicular to the plane, is a *vector quantity* since it has a specified magnitude and direction.

Magnitude. The magnitude of $\mathbf{M}_O$ is

$$M_O = Fd \tag{4–1}$$

where d is the *moment arm* or *perpendicular distance* from the axis at point O to the line of action of the force. Units of moment magnitude consist of force times distance, e.g., N · m or lb · ft.

Direction. The direction of $\mathbf{M}_O$ is defined by its *moment axis*, which is perpendicular to the plane that contains the force **F** and its moment arm d. The right-hand rule is used to establish the sense of direction of $\mathbf{M}_O$. According to this rule, the natural curl of the fingers of the right hand, as they are drawn towards the palm, represent the tendency for rotation caused by the moment. As this action is performed, the thumb of the right hand will give the directional sense of $\mathbf{M}_O$, Fig. 4–2*a*. Notice that the moment vector is represented three-dimensionally by a curl around an arrow. In two dimensions this vector is represented only by the curl as in Fig. 4–2*b*. Since in this case the moment will tend to cause a counterclockwise rotation, the moment vector is actually directed out of the page.

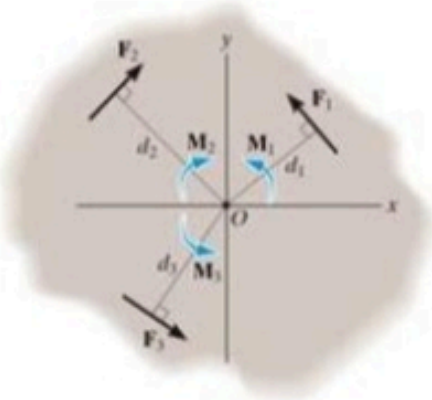

Fig. 4–3

Resultant Moment. For two-dimensional problems, where all the forces lie within the x–y plane, Fig. 4–3, the resultant moment $(\mathbf{M}_R)_O$ about point O (the z axis) can be determined by *finding the algebraic sum* of the moments caused by all the forces in the system. As a convention, we will generally consider *positive moments* as *counterclockwise* since they are directed along the positive z axis (out of the page). *Clockwise moments* will be *negative*. Doing this, the directional sense of each moment can be represented by a *plus or minus* sign. Using this sign convention, the resultant moment in Fig. 4–3 is therefore

$$\zeta + (M_R)_O = \Sigma Fd; \qquad (M_R)_O = F_1d_1 - F_2d_2 + F_3d_3$$

If the numerical result of this sum is a positive scalar, $(\mathbf{M}_R)_O$ will be a counterclockwise moment (out of the page); and if the result is negative, $(\mathbf{M}_R)_O$ will be a clockwise moment (into the page).

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EXAMPLE 4.1

For each case illustrated in Fig. 4–4, determine the moment of the force about point O.

SOLUTION (SCALAR ANALYSIS)

The line of action of each force is extended as a dashed line in order to establish the moment arm d. Also illustrated is the tendency of rotation of the member as caused by the force. Furthermore, the orbit of the force about O is shown as a colored curl. Thus,

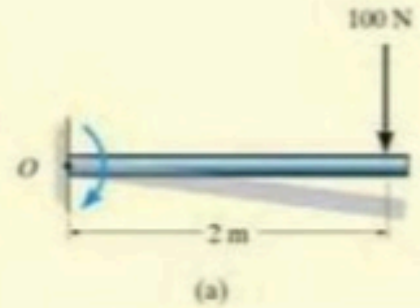
Fig. 4–4a $M_O = (100\text{ N})(2\text{ m}) = 200\text{ N}\cdot\text{m}$ ↷ *Ans.*

Fig. 4–4b $M_O = (50\text{ N})(0.75\text{ m}) = 37.5\text{ N}\cdot\text{m}$ ↷ *Ans.*

Fig. 4–4c $M_O = (40\text{ lb})(4\text{ ft} + 2\cos 30^\circ\text{ ft}) = 229\text{ lb}\cdot\text{ft}$ ↷ *Ans.*

Fig. 4–4d $M_O = (60\text{ lb})(1\sin 45^\circ\text{ ft}) = 42.4\text{ lb}\cdot\text{ft}$ ↶ *Ans.*

Fig. 4–4e $M_O = (7\text{ kN})(4\text{ m} - 1\text{ m}) = 21.0\text{ kN}\cdot\text{m}$ ↶ *Ans.*



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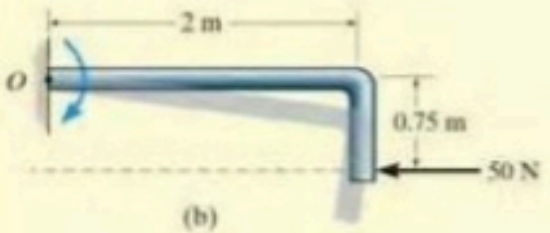

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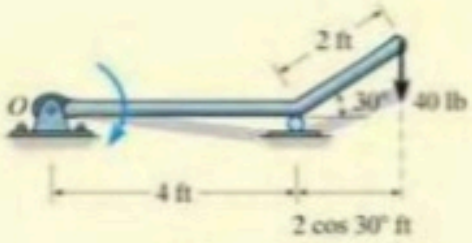

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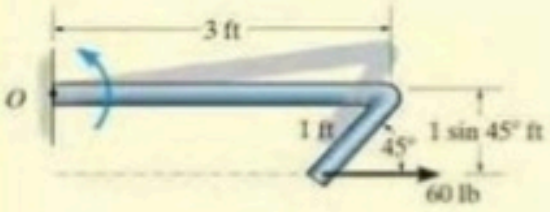

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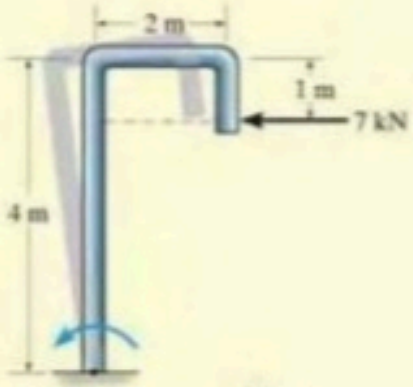

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Fig. 4–4

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EXAMPLE 4.2

Determine the resultant moment of the four forces acting on the rod shown in Fig. 4–5 about point O.

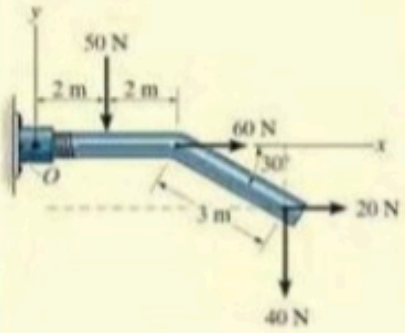


Fig. 4–5

SOLUTION

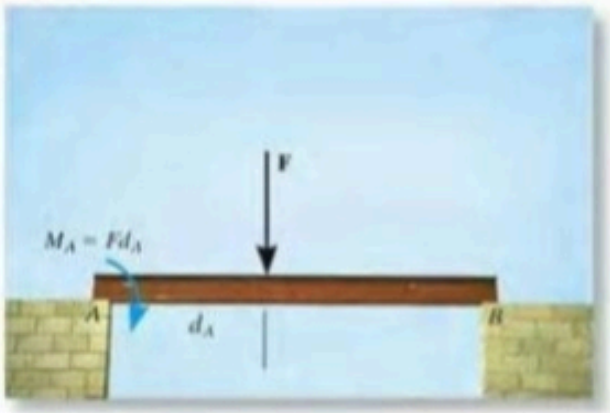
Assuming that positive moments act in the $+\mathbf{k}$ direction, i.e., counterclockwise, we have

$$\zeta + M_{R_O} = \Sigma Fd;$$

$$M_{R_O} = -50\,\text{N}(2\,\text{m}) + 60\,\text{N}(0) + 20\,\text{N}(3\sin 30^\circ\,\text{m}) - 40\,\text{N}(4\,\text{m} + 3\cos 30^\circ\,\text{m})$$

$$M_{R_O} = -334\,\text{N}\cdot\text{m} = 334\,\text{N}\cdot\text{m}\,\circlearrowright \qquad \textit{Ans.}$$

For this calculation, note how the moment-arm distances for the 20-N and 40-N forces are established from the extended (dashed) lines of action of each of these forces.



As illustrated by the example problems, the moment of a force does not always cause a rotation. For example, the force $\mathbf{F}$ tends to rotate the beam clockwise about its support at A with a moment $M_A = Fd_A$. The actual rotation would occur if the support at B were removed.

The ability to remove the nail will require the moment of $\mathbf{F}_H$ about point O to be larger than the moment of the force $\mathbf{F}_N$ about O that is needed to pull the nail out.

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4.2 Cross Product

The moment of a force will be formulated using Cartesian vectors in the next section. Before doing this, however, it is first necessary to expand our knowledge of vector algebra and introduce the cross-product method of vector multiplication.

The *cross product* of two vectors **A** and **B** yields the vector **C**, which is written

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} \tag{4–2}$$

and is read "**C** equals **A** cross **B**."

Magnitude. The *magnitude* of **C** is defined as the product of the magnitudes of **A** and **B** and the sine of the angle θ between their tails ($0° \leq \theta \leq 180°$). Thus, $C = AB \sin\theta$.

Direction. Vector **C** has a *direction* that is perpendicular to the plane containing **A** and **B** such that **C** is specified by the right-hand rule; i.e., curling the fingers of the right hand from vector **A** (cross) to vector **B**, the thumb points in the direction of **C**, as shown in Fig. 4–6.

Knowing both the magnitude and direction of **C**, we can write

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} = (AB \sin\theta)\mathbf{u}_C \tag{4–3}$$

where the scalar $AB \sin\theta$ defines the *magnitude* of **C** and the unit vector $\mathbf{u}_C$ defines the *direction* of **C**. The terms of Eq. 4–3 are illustrated graphically in Fig. 4–6.

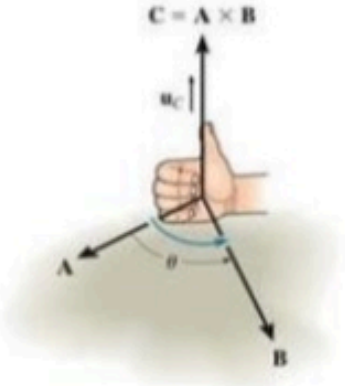

Fig. 4–6

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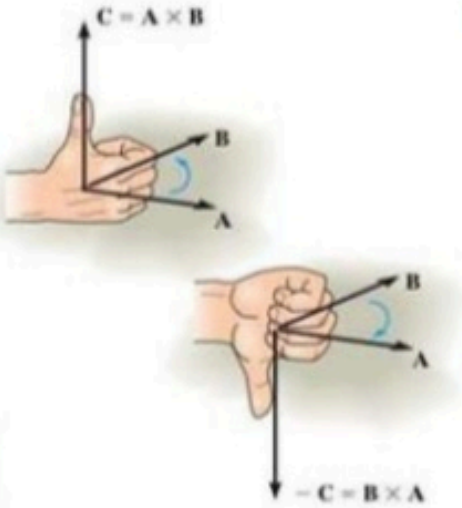

Fig. 4–7

Laws of Operation.

- The commutative law is *not* valid; i.e., $\mathbf{A} \times \mathbf{B} \neq \mathbf{B} \times \mathbf{A}$. Rather,

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$$

This is shown in Fig. 4–7 by using the right-hand rule. The cross product $\mathbf{B} \times \mathbf{A}$ yields a vector that has the same magnitude but acts in the opposite direction to $\mathbf{C}$; i.e., $\mathbf{B} \times \mathbf{A} = -\mathbf{C}$.

- If the cross product is multiplied by a scalar a, it obeys the associative law;

$$a(\mathbf{A} \times \mathbf{B}) = (a\mathbf{A}) \times \mathbf{B} = \mathbf{A} \times (a\mathbf{B}) = (\mathbf{A} \times \mathbf{B})a$$

This property is easily shown since the magnitude of the resultant vector ($|a|AB \sin\theta$) and its direction are the same in each case.

- The vector cross product also obeys the distributive law of addition,

$$\mathbf{A} \times (\mathbf{B} + \mathbf{D}) = (\mathbf{A} \times \mathbf{B}) + (\mathbf{A} \times \mathbf{D})$$

- The proof of this identity is left as an exercise (see Prob. 4–1). It is important to note that *proper order* of the cross products must be maintained, since they are not commutative.

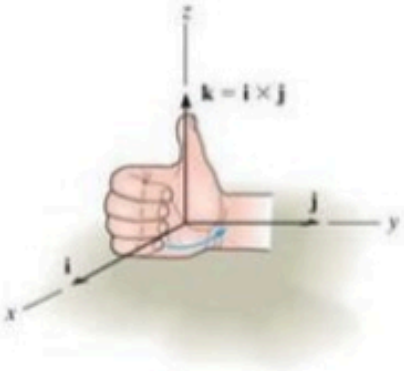

Fig. 4–8

Cartesian Vector Formulation. Equation 4–3 may be used to find the cross product of any pair of Cartesian unit vectors. For example, to find $\mathbf{i} \times \mathbf{j}$, the magnitude of the resultant vector is $(i)(j)(\sin 90°) = (1)(1)(1) = 1$, and its direction is determined using the right-hand rule. As shown in Fig. 4–8, the resultant vector points in the $+\mathbf{k}$ direction. Thus, $\mathbf{i} \times \mathbf{j} = (1)\mathbf{k}$. In a similar manner,

$$\begin{aligned} \mathbf{i} \times \mathbf{j} &= \mathbf{k} & \mathbf{i} \times \mathbf{k} &= -\mathbf{j} & \mathbf{i} \times \mathbf{i} &= \mathbf{0} \\ \mathbf{j} \times \mathbf{k} &= \mathbf{i} & \mathbf{j} \times \mathbf{i} &= -\mathbf{k} & \mathbf{j} \times \mathbf{j} &= \mathbf{0} \\ \mathbf{k} \times \mathbf{i} &= \mathbf{j} & \mathbf{k} \times \mathbf{j} &= -\mathbf{i} & \mathbf{k} \times \mathbf{k} &= \mathbf{0} \end{aligned}$$

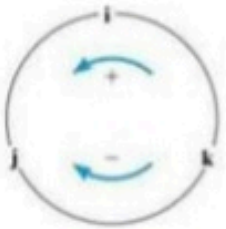

Fig. 4–9

These results should *not* be memorized; rather, it should be clearly understood how each is obtained by using the right-hand rule and the definition of the cross product. A simple scheme shown in Fig. 4–9 is helpful for obtaining the same results when the need arises. If the circle is constructed as shown, then "crossing" two unit vectors in a *counterclockwise* fashion around the circle yields the *positive* third unit vector; e.g., $\mathbf{k} \times \mathbf{i} = \mathbf{j}$. "Crossing" *clockwise*, a *negative* unit vector is obtained; e.g., $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$.

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Let us now consider the cross product of two general vectors **A** and **B** which are expressed in Cartesian vector form. We have

$$\begin{aligned}\mathbf{A} \times \mathbf{B} &= (A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}) \times (B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}) \\ &= A_xB_x(\mathbf{i} \times \mathbf{i}) + A_xB_y(\mathbf{i} \times \mathbf{j}) + A_xB_z(\mathbf{i} \times \mathbf{k}) \\ &+ A_yB_x(\mathbf{j} \times \mathbf{i}) + A_yB_y(\mathbf{j} \times \mathbf{j}) + A_yB_z(\mathbf{j} \times \mathbf{k}) \\ &+ A_zB_x(\mathbf{k} \times \mathbf{i}) + A_zB_y(\mathbf{k} \times \mathbf{j}) + A_zB_z(\mathbf{k} \times \mathbf{k})\end{aligned}$$

Carrying out the cross-product operations and combining terms yields

$$\mathbf{A} \times \mathbf{B} = (A_yB_z - A_zB_y)\mathbf{i} - (A_xB_z - A_zB_x)\mathbf{j} + (A_xB_y - A_yB_x)\mathbf{k} \quad (4\text{–}4)$$

This equation may also be written in a more compact determinant form as

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \quad (4\text{–}5)$$

Thus, to find the cross product of any two Cartesian vectors **A** and **B**, it is necessary to expand a determinant whose first row of elements consists of the unit vectors **i**, **j**, and **k** and whose second and third rows represent the x, y, z components of the two vectors **A** and **B**, respectively.*

*A determinant having three rows and three columns can be expanded using three minors, each of which is multiplied by one of the three terms in the first row. There are four elements in each minor, for example,

$$\begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix}$$

By *definition*, this determinant notation represents the terms $(A_{11}A_{22} - A_{12}A_{21})$, which is simply the product of the two elements intersected by the arrow slanting downward to the right $(A_{11}A_{22})$ *minus* the product of the two elements intersected by the arrow slanting downward to the left $(A_{12}A_{21})$. For a 3×3 determinant, such as Eq. 4–5, the three minors can be generated in accordance with the following scheme:

For element **i**:
$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \mathbf{i}(A_yB_z - A_zB_y)$$

For element **j**:
$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = -\mathbf{j}(A_xB_z - A_zB_x)$$

Remember the negative sign

For element **k**:
$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \mathbf{k}(A_xB_y - A_yB_x)$$

Adding the results and noting that the **j** element *must include the minus sign* yields the expanded form of $\mathbf{A} \times \mathbf{B}$ given by Eq. 4–4.

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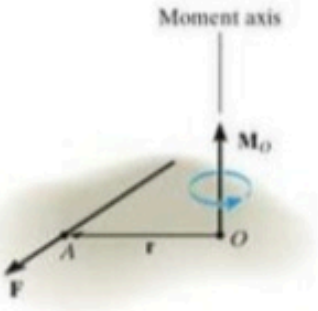

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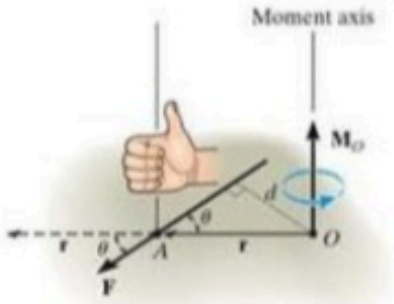

(b)

Fig. 4–10

4.3 Moment of a Force—Vector Formulation

The moment of a force **F** about point O, or actually about the moment axis passing through O and perpendicular to the plane containing O and **F**, Fig. 4–10*a*, can be expressed using the vector cross product, namely,

$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F} \tag{4–6}$$

Here **r** represents a position vector directed *from* O to *any point* on the line of action of **F**. We will now show that indeed the moment $\mathbf{M}_O$, when determined by this cross product, has the proper magnitude and direction.

Magnitude. The magnitude of the cross product is defined from Eq. 4–3 as $M_O = rF \sin\theta$, where the angle θ is measured between the *tails* of **r** and **F**. To establish this angle, **r** must be treated as a sliding vector so that θ can be constructed properly, Fig. 4–10*b*. Since the moment arm $d = r \sin\theta$, then

$$M_O = rF \sin\theta = F(r \sin\theta) = Fd$$

which agrees with Eq. 4–1.

Direction. The direction and sense of $\mathbf{M}_O$ in Eq. 4–6 are determined by the right-hand rule as it applies to the cross product. Thus, sliding **r** to the dashed position and curling the right-hand fingers from **r** toward **F**, "**r** cross **F**," the thumb is directed upward or perpendicular to the plane containing **r** and **F** and this is in the *same direction* as $\mathbf{M}_O$, the moment of the force about point O, Fig. 4–10*b*. Note that the "curl" of the fingers, like the curl around the moment vector, indicates the sense of rotation caused by the force. Since the cross product does not obey the commutative law, the order of $\mathbf{r} \times \mathbf{F}$ must be maintained to produce the correct sense of direction for $\mathbf{M}_O$.

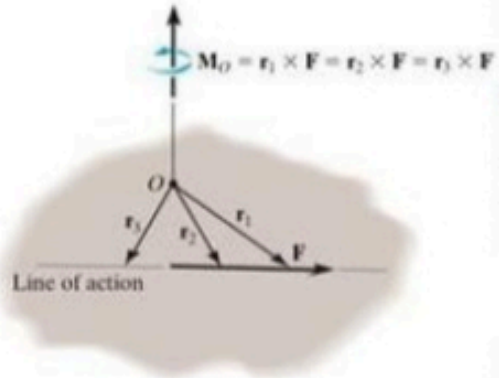

Fig. 4–11

Principle of Transmissibility. The cross product operation is often used in three dimensions since the perpendicular distance or moment arm from point O to the line of action of the force is not needed. In other words, we can use any position vector **r** measured from point O to any point on the line of action of the force **F**, Fig. 4–11. Thus,

$$\mathbf{M}_O = \mathbf{r}_1 \times \mathbf{F} = \mathbf{r}_2 \times \mathbf{F} = \mathbf{r}_3 \times \mathbf{F}$$

Since **F** can be applied at any point along its line of action and still create this *same moment* about point O, then **F** can be considered a *sliding vector*. This property is called the *principle of transmissibility* of a force.

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Cartesian Vector Formulation. If we establish x, y, z coordinate axes, then the position vector $\mathbf{r}$ and force $\mathbf{F}$ can be expressed as Cartesian vectors, Fig. 4–12*a*. Applying Eq. 4–5 we have

$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix} \tag{4–7}$$

where

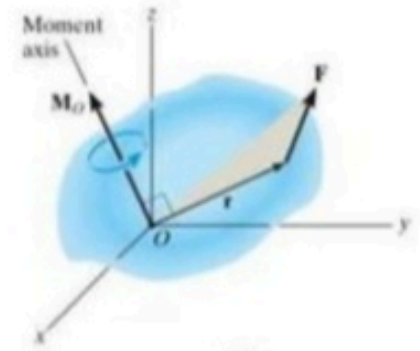
r_x, r_y, r_z represent the x, y, z components of the position vector drawn from point O to *any point* on the line of action of the force

F_x, F_y, F_z represent the x, y, z components of the force vector

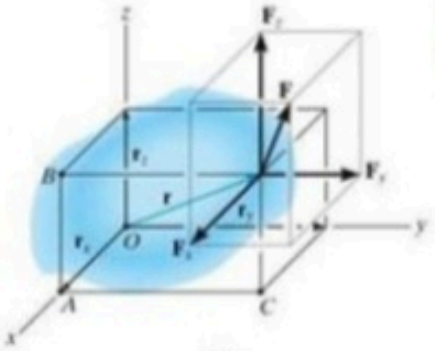
If the determinant is expanded, then like Eq. 4–4 we have

$$\mathbf{M}_O = (r_y F_z - r_z F_y)\mathbf{i} - (r_x F_z - r_z F_x)\mathbf{j} + (r_x F_y - r_y F_x)\mathbf{k} \tag{4–8}$$

The physical meaning of these three moment components becomes evident by studying Fig. 4–12*b*. For example, the $\mathbf{i}$ component of $\mathbf{M}_O$ can be determined from the moments of $\mathbf{F}_x$, $\mathbf{F}_y$, and $\mathbf{F}_z$ about the x axis. The component $\mathbf{F}_x$ does *not* create a moment or tendency to cause turning about the x axis since this force is *parallel* to the x axis. The line of action of $\mathbf{F}_y$ passes through point B, and so the magnitude of the moment of $\mathbf{F}_y$ about point A on the x axis is $r_z F_y$. By the right-hand rule this component acts in the *negative* $\mathbf{i}$ direction. Likewise, $\mathbf{F}_z$ passes through point C and so it contributes a moment component of $r_y F_z \mathbf{i}$ about the axis. Thus, $(M_O)_x = (r_y F_z - r_z F_y)$ as shown in Eq. 4–8. As an exercise, establish the $\mathbf{j}$ and $\mathbf{k}$ components of $\mathbf{M}_O$ in this manner and show that indeed the expanded form of the determinant, Eq. 4–8, represents the moment of $\mathbf{F}$ about point O. Once $\mathbf{M}_O$ is determined, realize that it will always be *perpendicular* to the shaded plane containing vectors $\mathbf{r}$ and $\mathbf{F}$, Fig. 4–12*a*.



(a)



(b)

Fig. 4–12

Resultant Moment of a System of Forces. If a body is acted upon by a system of forces, Fig. 4–13, the resultant moment of the forces about point O can be determined by vector addition of the moment of each force. This resultant can be written symbolically as

$$\mathbf{M}_{R_O} = \Sigma(\mathbf{r} \times \mathbf{F}) \tag{4–9}$$

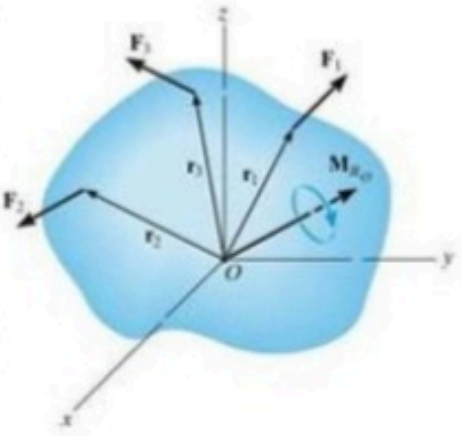
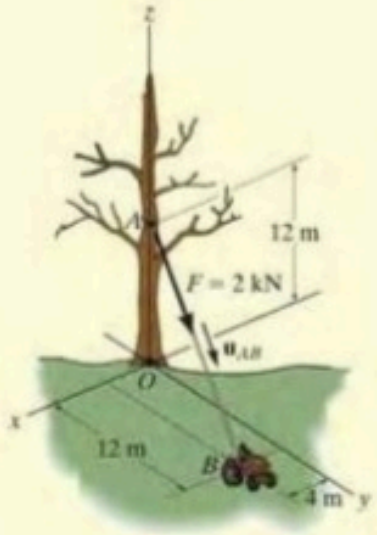


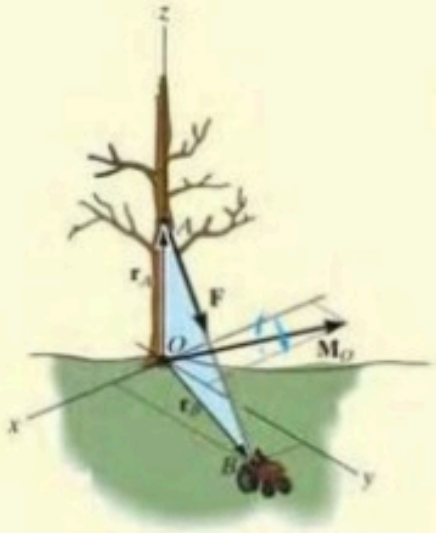
Fig. 4–13

Easy PDF

EXAMPLE 4.3



(a)



(b)

Fig. 4–14

Determine the moment produced by the force **F** in Fig. 4–14*a* about point *O*. Express the result as a Cartesian vector.

SOLUTION

As shown in Fig. 4–14*a*, either $\mathbf{r}_A$ or $\mathbf{r}_B$ can be used to determine the moment about point *O*. These position vectors are

$$\mathbf{r}_A = \{12\mathbf{k}\}\ \text{m} \quad \text{and} \quad \mathbf{r}_B = \{4\mathbf{i} + 12\mathbf{j}\}\ \text{m}$$

Force **F** expressed as a Cartesian vector is

$$\mathbf{F} = F\mathbf{u}_{AB} = 2\ \text{kN}\left[\frac{\{4\mathbf{i} + 12\mathbf{j} - 12\mathbf{k}\}\ \text{m}}{\sqrt{(4\ \text{m})^2 + (12\ \text{m})^2 + (-12\ \text{m})^2}}\right]$$

$$= \{0.4588\mathbf{i} + 1.376\mathbf{j} - 1.376\mathbf{k}\}\ \text{kN}$$

Thus

$$\mathbf{M}_O = \mathbf{r}_A \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 12 \\ 0.4588 & 1.376 & -1.376 \end{vmatrix}$$

$$= [0(-1.376) - 12(1.376)]\mathbf{i} - [0(-1.376) - 12(0.4588)]\,\mathbf{j} + [0(1.376) - 0(0.4588)]\mathbf{k}$$

$$= \{-16.5\mathbf{i} + 5.51\mathbf{j}\}\ \text{kN·m} \qquad \textit{Ans.}$$

or

$$\mathbf{M}_O = \mathbf{r}_B \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 12 & 0 \\ 0.4588 & 1.376 & -1.376 \end{vmatrix}$$

$$= [12(-1.376) - 0(1.376)]\mathbf{i} - [4(-1.376) - 0(0.4588)]\mathbf{j} + [4(1.376) - 12(0.4588)]\mathbf{k}$$

$$= \{-16.5\mathbf{i} + 5.51\mathbf{j}\}\ \text{kN·m} \qquad \textit{Ans.}$$

NOTE: As shown in Fig. 4–14*b*, $\mathbf{M}_O$ acts perpendicular to the plane that contains **F**, $\mathbf{r}_A$, and $\mathbf{r}_B$. Had this problem been worked using $M_O = Fd$, notice the difficulty that would arise in obtaining the moment arm *d*.

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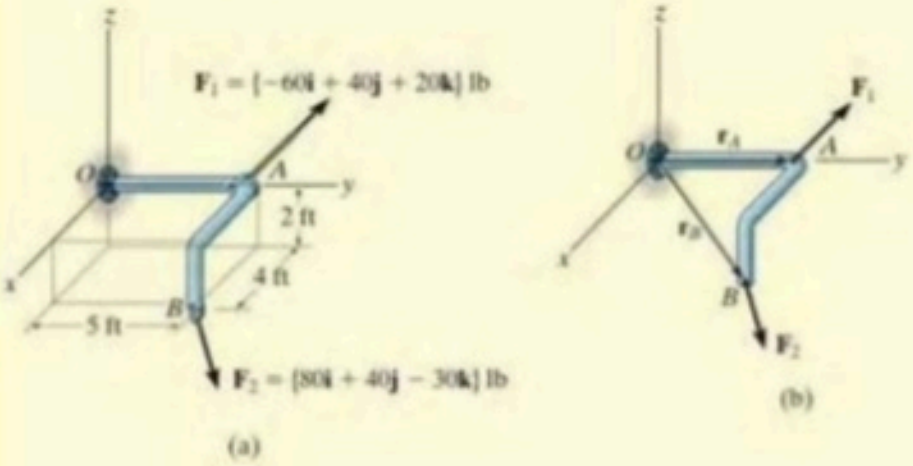
EXAMPLE 4.4

Two forces act on the rod shown in Fig. 4–15*a*. Determine the resultant moment they create about the flange at *O*. Express the result as a Cartesian vector.



SOLUTION

Position vectors are directed from point *O* to each force as shown in Fig. 4–15*b*. These vectors are

$$\mathbf{r}_A = \{5\mathbf{j}\}\ \text{ft}$$

$$\mathbf{r}_B = \{4\mathbf{i} + 5\mathbf{j} - 2\mathbf{k}\}\ \text{ft}$$

The resultant moment about *O* is therefore

$$\mathbf{M}_{R_O} = \Sigma(\mathbf{r} \times \mathbf{F})$$

$$= \mathbf{r}_A \times \mathbf{F}_1 + \mathbf{r}_B \times \mathbf{F}_2$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 5 & 0 \\ -60 & 40 & 20 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 5 & -2 \\ 80 & 40 & -30 \end{vmatrix}$$

$$= [5(20) - 0(40)]\mathbf{i} - [0]\mathbf{j} + [0(40) - (5)(-60)]\mathbf{k}$$

$$+ [5(-30) - (-2)(40)]\mathbf{i} - [4(-30) - (-2)(80)]\mathbf{j} + [4(40) - 5(80)]\mathbf{k}$$

$$= \{30\mathbf{i} - 40\mathbf{j} + 60\mathbf{k}\}\ \text{lb}\cdot\text{ft} \qquad \textit{Ans.}$$

NOTE: This result is shown in Fig. 4–15*c*. The coordinate direction angles were determined from the unit vector for $\mathbf{M}_{R_O}$. Realize that the two forces tend to cause the rod to rotate about the moment axis in the manner shown by the curl indicated on the moment vector.

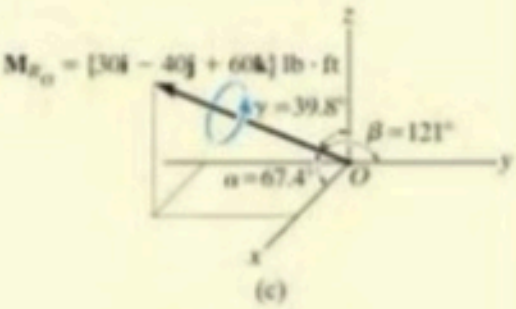


Fig. 4–15

Easy PDF

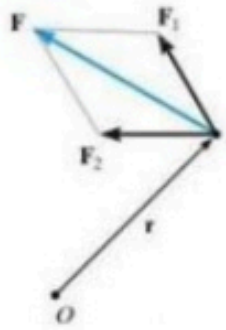


Fig 4–16

4.4 Principle of Moments

A concept often used in mechanics is the *principle of moments*, which is sometimes referred to as *Varignon's theorem* since it was originally developed by the French mathematician Varignon (1654–1722). It states that *the moment of a force about a point is equal to the sum of the moments of the components of the force about the point.* This theorem can be proven easily using the vector cross product since the cross product obeys the *distributive law*. For example, consider the moments of the force **F** and two of its components about point O. Fig. 4–16. Since $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$ we have

$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F} = \mathbf{r} \times (\mathbf{F}_1 + \mathbf{F}_2) = \mathbf{r} \times \mathbf{F}_1 + \mathbf{r} \times \mathbf{F}_2$$

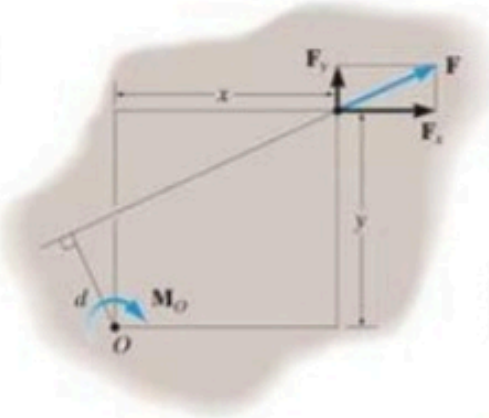


Fig. 4–17

For two-dimensional problems, Fig. 4–17, we can use the principle of moments by resolving the force into its rectangular components and then determine the moment using a scalar analysis. Thus,

$$M_O = F_x y - F_y x$$

This method is generally easier than finding the same moment using $M_O = Fd$.

Important Points

- The moment of a force creates the tendency of a body to turn about an axis passing through a specific point O.
- Using the right-hand rule, the sense of rotation is indicated by the curl of the fingers, and the thumb is directed along the moment axis, or line of action of the moment.
- The magnitude of the moment is determined from $M_O = Fd$, where d is called the moment arm, which represents the perpendicular or shortest distance from point O to the line of action of the force.
- In three dimensions the vector cross product is used to determine the moment, i.e., $\mathbf{M}_O = \mathbf{r} \times \mathbf{F}$. Remember that **r** is directed *from* point O *to any point* on the line of action of **F**.
- The principle of moments states that the moment of a force about a point is equal to the sum of the moments of the force's components about the point. This is a very convenient method to use in two dimensions.

The moment of the applied force **F** about point O is easy to determine if we use the principle of moments. It is simply $M_O = F_x d$.

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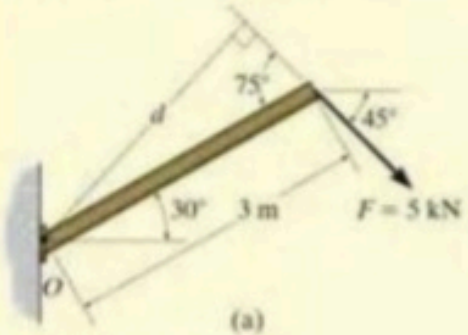
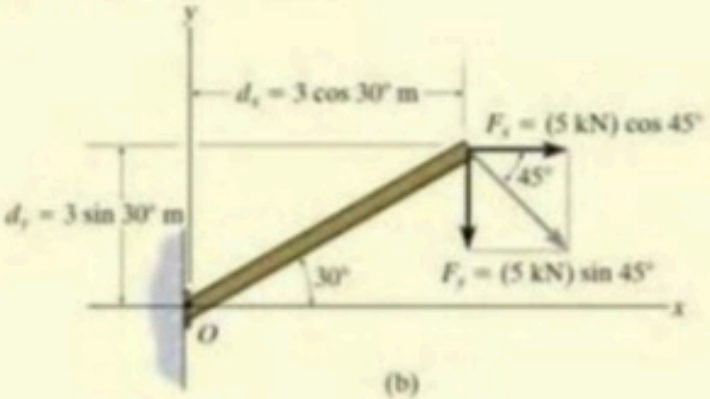
EXAMPLE 4.5

Determine the moment of the force in Fig. 4–18*a* about point *O*.



(a)



(b)

SOLUTION I

The moment arm d in Fig. 4–18*a* can be found from trigonometry.

$$d = (3\text{ m})\sin 75° = 2.898\text{ m}$$

Thus,

$$M_O = Fd = (5\text{kN})(2.898\text{ m}) = 14.5\text{ kN}\cdot\text{m}\circlearrowright \qquad \textit{Ans.}$$

Since the force tends to rotate or orbit clockwise about point *O*, the moment is directed into the page.

SOLUTION II

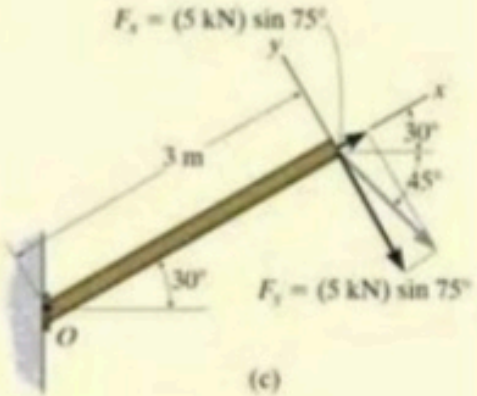
The x and y components of the force are indicated in Fig. 4–18*b*. Considering counterclockwise moments as positive, and applying the principle of moments, we have

$$\begin{aligned}\zeta + M_O &= -F_x d_y - F_y d_x \\ &= -(5\cos 45°\text{ kN})(3\sin 30°\text{ m}) - (5\sin 45°\text{ kN})(3\cos 30°\text{ m}) \\ &= -14.5\text{ kN}\cdot\text{m} = 14.5\text{ kN}\cdot\text{m}\circlearrowright\end{aligned} \qquad \textit{Ans.}$$

SOLUTION III

The x and y axes can be set parallel and perpendicular to the rod's axis as shown in Fig. 4-18*c*. Here $\mathbf{F}_x$ produces no moment about point *O* since its line of action passes through this point. Therefore,

$$\begin{aligned}\zeta + M_O &= -F_y d_x \\ &= -(5\sin 75°\text{ kN})(3\text{ m}) \\ &= -14.5\text{ kN}\cdot\text{m} = 14.5\text{ kN}\cdot\text{m}\circlearrowright\end{aligned} \qquad \textit{Ans.}$$



(c)

Fig. 4–18

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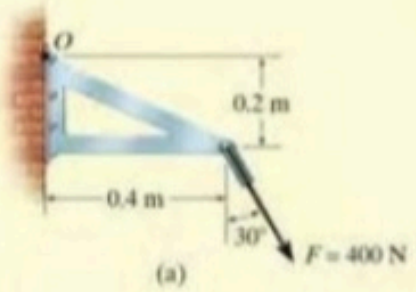
EXAMPLE 4.6

Force **F** acts at the end of the angle bracket shown in Fig. 4–19*a*. Determine the moment of the force about point *O*.



(a)

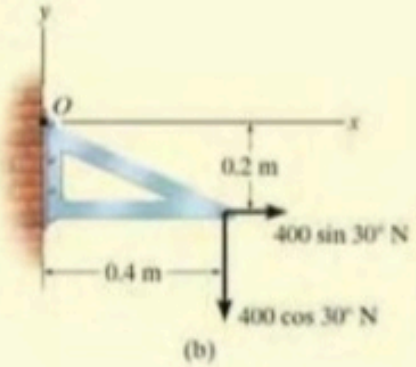
SOLUTION I *(SCALAR ANALYSIS)*

The force is resolved into its x and y components as shown in Fig. 4–19*b*, then

$$\zeta + M_O = 400 \sin 30^\circ \text{ N}(0.2 \text{ m}) - 400 \cos 30^\circ \text{ N}(0.4 \text{ m})$$
$$= -98.6 \text{ N} \cdot \text{m} = 98.6 \text{ N} \cdot \text{m} \,\zeta$$

or

$$\mathbf{M}_O = \{-98.6\mathbf{k}\} \text{ N} \cdot \text{m} \qquad \textit{Ans.}$$



(b)

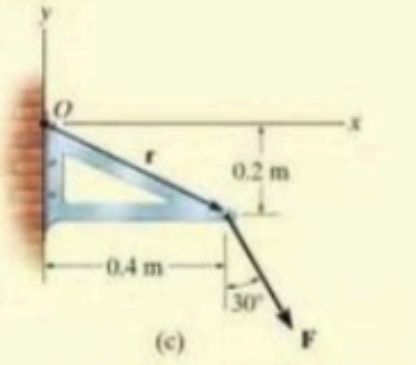
SOLUTION II *(VECTOR ANALYSIS)*

Using a Cartesian vector approach, the force and position vectors shown in Fig. 4–19*c* are

$$\mathbf{r} = \{0.4\mathbf{i} - 0.2\mathbf{j}\} \text{ m}$$
$$\mathbf{F} = \{400 \sin 30^\circ \mathbf{i} - 400 \cos 30^\circ \mathbf{j}\} \text{ N}$$
$$= \{200.0\mathbf{i} - 346.4\mathbf{j}\} \text{ N}$$

The moment is therefore

$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.4 & -0.2 & 0 \\ 200.0 & -346.4 & 0 \end{vmatrix}$$
$$= 0\mathbf{i} - 0\mathbf{j} + [0.4(-346.4) - (-0.2)(200.0)]\mathbf{k}$$
$$= \{-98.6\mathbf{k}\} \text{ N} \cdot \text{m} \qquad \textit{Ans.}$$



(c)

Fig. 4–19

NOTE: It is seen that the scalar analysis (Solution I) provides a more *convenient method* for analysis than Solution II since the direction of the moment and the moment arm for each component force are easy to establish. Hence, this method is generally recommended for solving problems displayed in two dimensions, whereas a Cartesian vector analysis is generally recommended only for solving three-dimensional problems.

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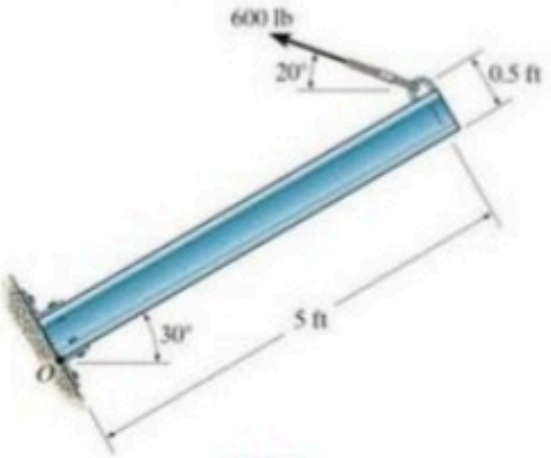
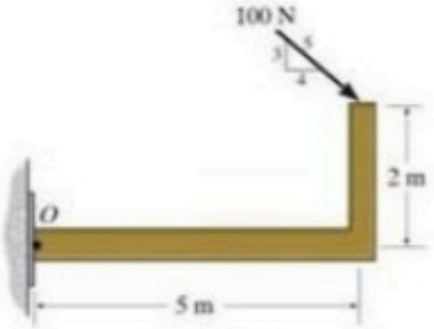
FUNDAMENTAL PROBLEMS

F4–1. Determine the moment of the force about point O.



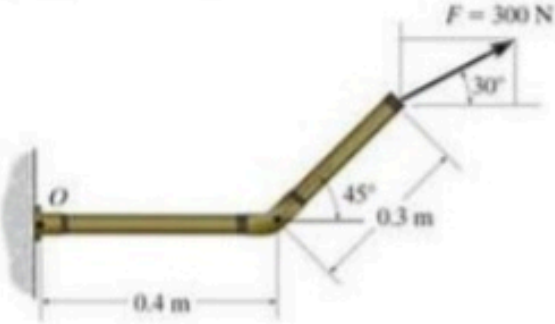
F4–1

F4–2. Determine the moment of the force about point O.



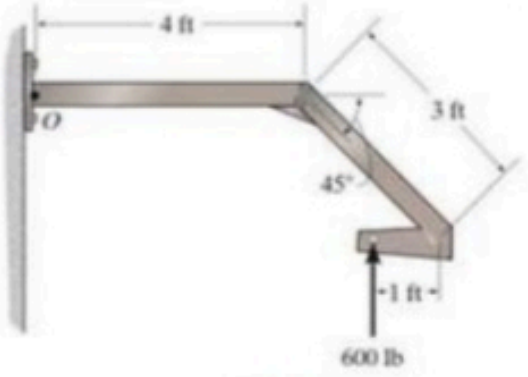
F4–2

F4–3. Determine the moment of the force about point O.



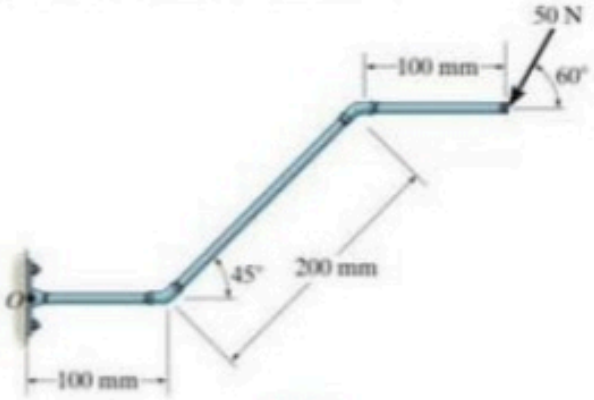
F4–3

F4–4. Determine the moment of the force about point O.



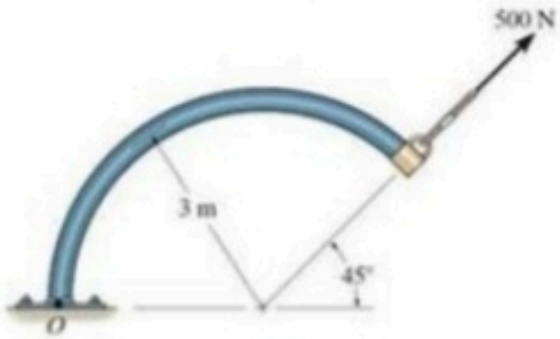
F4–4

F4–5. Determine the moment of the force about point O. Neglect the thickness of the member.



F4–5

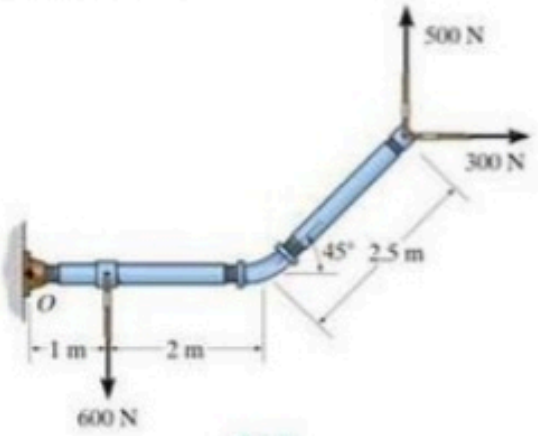
F4–6. Determine the moment of the force about point O.



F4–6

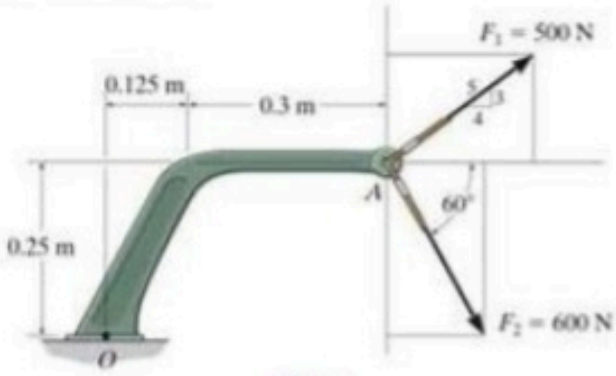
Easy PDF

F4–7. Determine the resultant moment produced by the forces about point *O*.



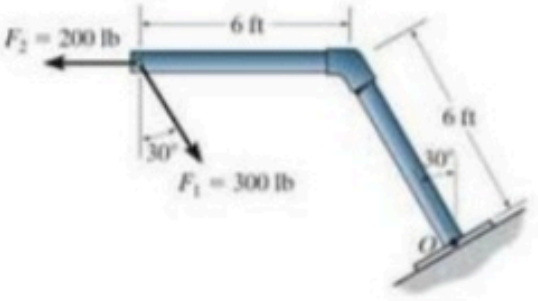
F4–7

F4–8. Determine the resultant moment produced by the forces about point *O*.



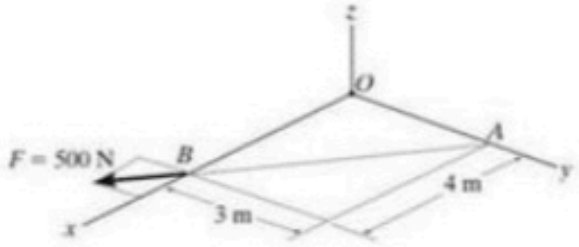
F4–8

F4–9. Determine the resultant moment produced by the forces about point *O*.



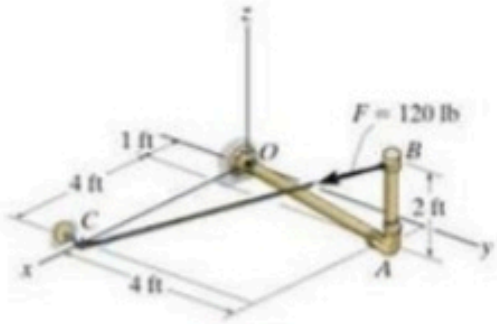
F4–9

F4–10. Determine the moment of force **F** about point *O*. Express the result as a Cartesian vector.



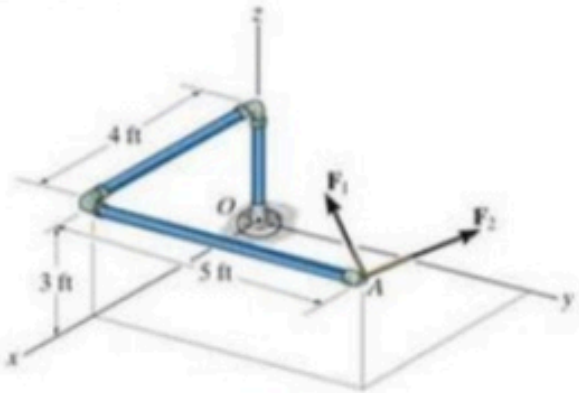
F 4–10

F 4–11. Determine the moment of force **F** about point *O*. Express the result as a Cartesian vector.



F4–11

F4–12. If $\mathbf{F}_1 = \{100\mathbf{i} - 120\mathbf{j} + 75\mathbf{k}\}$ lb and $\mathbf{F}_2 = \{-200\mathbf{i} + 250\mathbf{j} + 100\mathbf{k}\}$ lb, determine the resultant moment produced by these forces about point *O*. Express the result as a Cartesian vector.



F4–12

Easy PDF

PROBLEMS

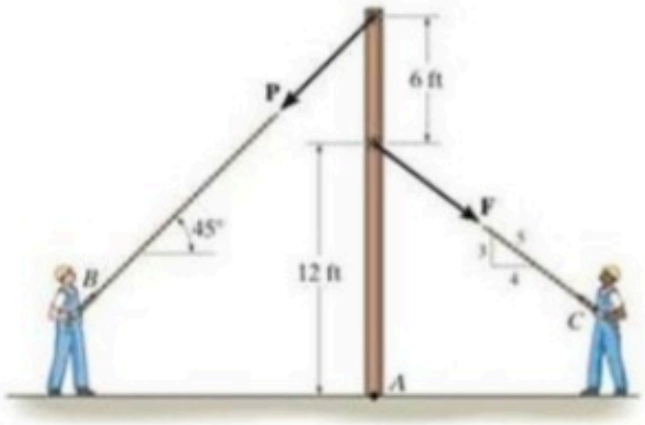
•4–1. If **A**, **B**, and **D** are given vectors, prove the distributive law for the vector cross product, i.e., $\mathbf{A} \times (\mathbf{B} + \mathbf{D}) = (\mathbf{A} \times \mathbf{B}) + (\mathbf{A} \times \mathbf{D})$.

4–2. Prove the triple scalar product identity $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = \mathbf{A} \times \mathbf{B} \cdot \mathbf{C}$.

4–3. Given the three nonzero vectors **A**, **B**, and **C**, show that if $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = 0$, the three vectors *must* lie in the same plane.

***4–4.** Two men exert forces of $F = 80$ lb and $P = 50$ lb on the ropes. Determine the moment of each force about A. Which way will the pole rotate, clockwise or counterclockwise?

•4–5. If the man at B exerts a force of $P = 30$ lb on his rope, determine the magnitude of the force **F** the man at C must exert to prevent the pole from rotating, i.e., so the resultant moment about A of both forces is zero.



Probs. 4–4/5

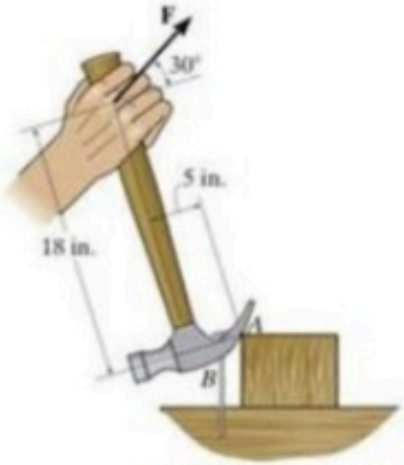
4–6. If $\theta = 45°$, determine the moment produced by the 4-kN force about point A.

4–7. If the moment produced by the 4-kN force about point A is 10 kN · m clockwise, determine the angle θ, where $0° \leq \theta \leq 90°$.

Probs. 4–6/7

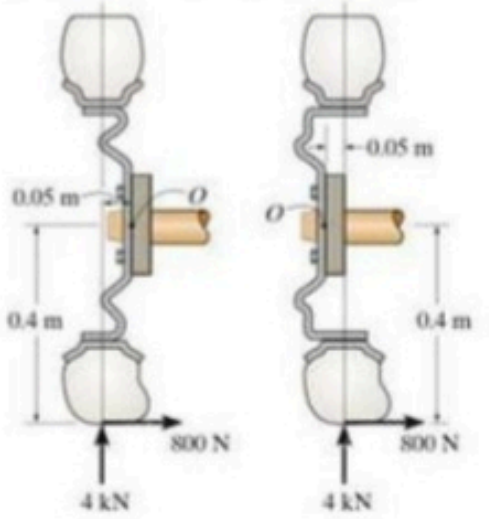
***4–8.** The handle of the hammer is subjected to the force of $F = 20$ lb. Determine the moment of this force about the point A.

•4–9. In order to pull out the nail at B, the force **F** exerted on the handle of the hammer must produce a clockwise moment of 500 lb · in. about point A. Determine the required magnitude of force **F**.



Probs. 4–8/9

4–10. The hub of the wheel can be attached to the axle either with negative offset (left) or with positive offset (right). If the tire is subjected to both a normal and radial load as shown, determine the resultant moment of these loads about point O on the axle for both cases.



Prob. 4–10

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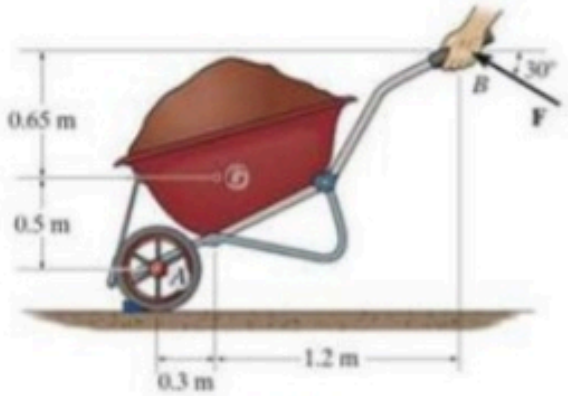
4–34. In order to hold the wheelbarrow in the position shown, force **F** must produce a counterclockwise moment of 200 N · m about the axle at *A*. Determine the required magnitude of force **F**.

4–35. The wheelbarrow and its contents have a mass of 50 kg and a center of mass at *G*. If the resultant moment produced by force **F** and the weight about point *A* is to be zero, determine the required magnitude of force **F**.

***4–36.** The wheelbarrow and its contents have a center of mass at *G*. If $F = 100$ N and the resultant moment produced by force **F** and the weight about the axle at *A* is zero, determine the mass of the wheelbarrow and its contents.

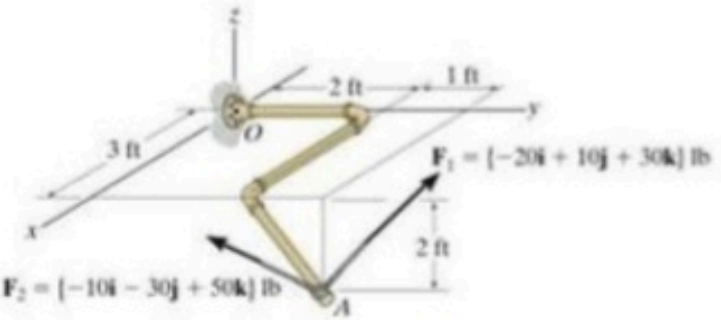


Prob. 4–34/35/36

•**4–37.** Determine the moment produced by $\mathbf{F}_1$ about point *O*. Express the result as a Cartesian vector.

4–38. Determine the moment produced by $\mathbf{F}_2$ about point *O*. Express the result as a Cartesian vector.

4–39. Determine the resultant moment produced by the two forces about point *O*. Express the result as a Cartesian vector.

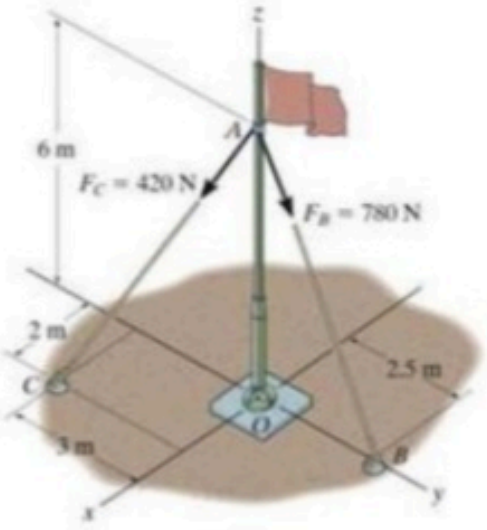


Probs. 4–37/38/39

***4–40.** Determine the moment produced by force $\mathbf{F}_B$ about point *O*. Express the result as a Cartesian vector.

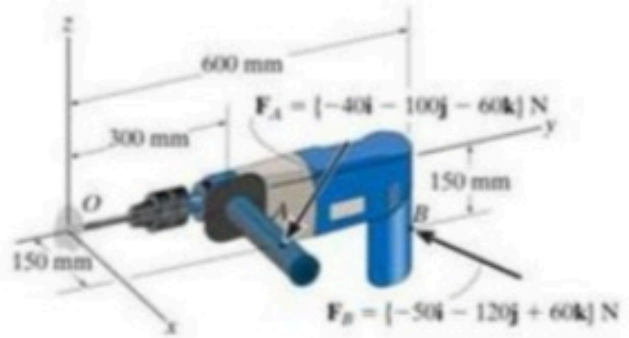
•**4–41.** Determine the moment produced by force $\mathbf{F}_C$ about point *O*. Express the result as a Cartesian vector.

4–42. Determine the resultant moment produced by forces $\mathbf{F}_B$ and $\mathbf{F}_C$ about point *O*. Express the result as a Cartesian vector.



Probs. 4–40/41/42

4–43. Determine the moment produced by each force about point *O* located on the drill bit. Express the results as Cartesian vectors.



Prob. 4–43

***4–44.** A force of $\mathbf{F} = \{6\mathbf{i} - 2\mathbf{j} + 1\mathbf{k}\}$ kN produces a moment of $\mathbf{M}_O = \{4\mathbf{i} + 5\mathbf{j} - 14\mathbf{k}\}$ kN · m about the origin of coordinates, point *O*. If the force acts at a point having an *x* coordinate of $x = 1$ m, determine the *y* and *z* coordinates.

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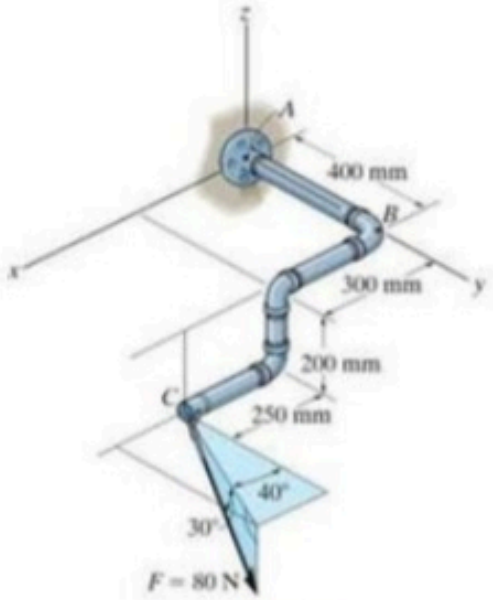
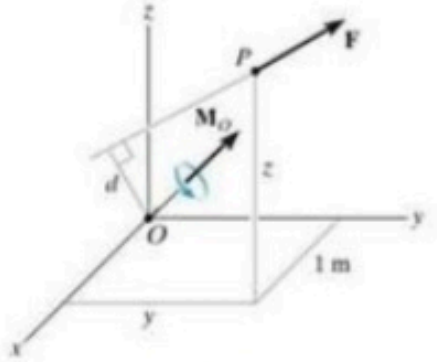
•4–45. The pipe assembly is subjected to the 80-N force. Determine the moment of this force about point *A*.

4–46. The pipe assembly is subjected to the 80-N force. Determine the moment of this force about point *B*.



Probs. 4–45/46

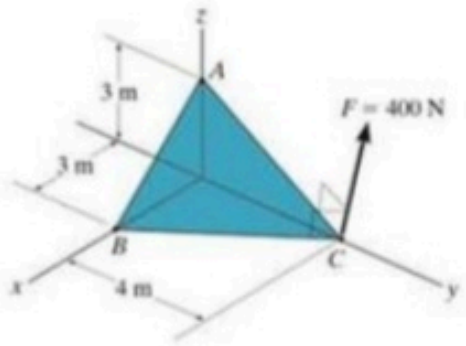
4–47. The force $\mathbf{F} = \{6\mathbf{i} + 8\mathbf{j} + 10\mathbf{k}\}$ N creates a moment about point *O* of $\mathbf{M}_O = \{-14\mathbf{i} + 8\mathbf{j} + 2\mathbf{k}\}$ N·m. If the force passes through a point having an *x* coordinate of 1 m, determine the *y* and *z* coordinates of the point. Also, realizing that $M_O = Fd$, determine the perpendicular distance *d* from point *O* to the line of action of **F**.



Prob. 4–47

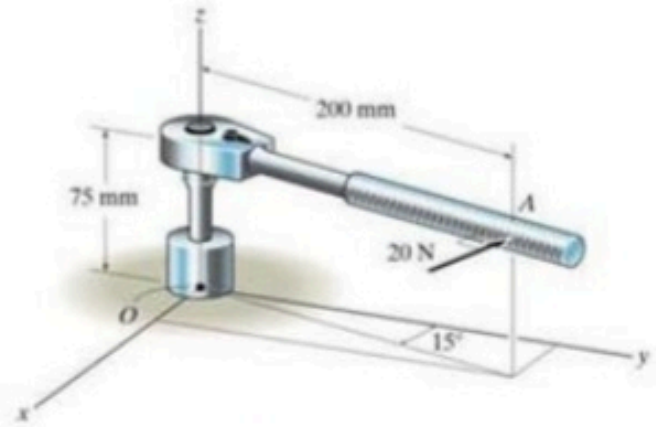
•4–48. Force **F** acts perpendicular to the inclined plane. Determine the moment produced by **F** about point *A*. Express the result as a Cartesian vector.

•4–49. Force **F** acts perpendicular to the inclined plane. Determine the moment produced by **F** about point *B*. Express the result as a Cartesian vector.



Probs. 4–48/49

4–50. A 20-N horizontal force is applied perpendicular to the handle of the socket wrench. Determine the magnitude and the coordinate direction angles of the moment created by this force about point *O*.



Prob. 4–50

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