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Mathematics (II)

By

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II. In the exercises below, find:

a. $\mathbf{v} \cdot \mathbf{u}$, $|\mathbf{v}|$, $|\mathbf{u}|$

b. the cosine of the angle between $\mathbf{v}$ and $\mathbf{u}$

1. $\mathbf{v} = 10\mathbf{i} + 11\mathbf{j} - 2\mathbf{k}$, $\mathbf{u} = 3\mathbf{j} + 4\mathbf{k}$

2. $\mathbf{v} = 2\mathbf{i} + 10\mathbf{j} - 11\mathbf{k}$, $\mathbf{u} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$

3. $\mathbf{v} = 5\mathbf{j} - 3\mathbf{k}$, $\mathbf{u} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

4. $\mathbf{v} = -\mathbf{i} + \mathbf{j}$, $\mathbf{u} = \sqrt{2}\mathbf{i} + \sqrt{3}\mathbf{j} + 2\mathbf{k}$

Solution:

	$\mathbf{v} \cdot \mathbf{u}$	$ \mathbf{v} $	$ \mathbf{u} $	$\cos \theta$
1.	25	15	5	$\frac{1}{3}$
2.	13	15	3	$\frac{13}{45}$
3.	2	$\sqrt{34}$	$\sqrt{3}$	$\frac{2}{\sqrt{3}\sqrt{34}}$
4.	$\sqrt{3} - \sqrt{2}$	$\sqrt{2}$	3	$\frac{\sqrt{3} - \sqrt{2}}{3\sqrt{2}}$

I. Find the length and direction (when defined) of $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$

Solution:

1. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & 0 \\ 0 & -3 & 0 \end{vmatrix} = -6(\mathbf{k}) \Rightarrow \text{length} = 6 \text{ and the direction is } -\mathbf{k}$

$\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = 6(\mathbf{k}) \Rightarrow \text{length} = 6 \text{ and the direction is } \mathbf{k}$

2. $\mathbf{u} \times \mathbf{v} = (\mathbf{i} \times \mathbf{j}) \times (\mathbf{j} \times \mathbf{k}) = \mathbf{k} \times \mathbf{i} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = \mathbf{j} \Rightarrow \text{length} = 1 \text{ and the direction is } \mathbf{j}$

$\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = -\mathbf{j} \Rightarrow \text{length} = 1 \text{ and the direction is } -\mathbf{j}$

3. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -8 & -2 & -4 \\ 2 & 2 & 1 \end{vmatrix} = 6\mathbf{i} - 12\mathbf{k} \Rightarrow \text{length} = 6\sqrt{5} \text{ and the direction is } \frac{1}{\sqrt{5}}\mathbf{i} - \frac{2}{\sqrt{5}}\mathbf{k}$

$\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = -(6\mathbf{i} - 12\mathbf{k}) \Rightarrow \text{length} = 6\sqrt{5} \text{ and the direction is } -\frac{1}{\sqrt{5}}\mathbf{i} + \frac{2}{\sqrt{5}}\mathbf{k}$

4. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{3}{2} & -\frac{1}{2} & 1 \\ 1 & 1 & 2 \end{vmatrix} = -2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k} \Rightarrow \text{length} = 2\sqrt{3} \text{ and the direction is } -\frac{1}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$

$\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = -(-2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}) \Rightarrow \text{length} = 2\sqrt{3} \text{ and the direction is } \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k}$

1. $\mathbf{u} = 2\mathbf{i}$, $\mathbf{v} = -3\mathbf{j}$

2. $\mathbf{u} = \mathbf{i} \times \mathbf{j}$, $\mathbf{v} = \mathbf{j} \times \mathbf{k}$

3. $\mathbf{u} = -8\mathbf{i} - 2\mathbf{j} - 4\mathbf{k}$, $\mathbf{v} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$

4. $\mathbf{u} = \frac{3}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} + \mathbf{k}$, $\mathbf{v} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$

III. Find the angles between the vectors

1. $\mathbf{u} = \sqrt{3}\mathbf{i} - 7\mathbf{j}$, $\mathbf{v} = \sqrt{3}\mathbf{i} + \mathbf{j} - 2\mathbf{k}$
2. $\mathbf{u} = \mathbf{i} + \sqrt{2}\mathbf{j} - \sqrt{2}\mathbf{k}$, $\mathbf{v} = -\mathbf{i} + \mathbf{j} + \mathbf{k}$

Solution:

$$1. \quad \theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\right) = \cos^{-1}\left(\frac{(\sqrt{3})(\sqrt{3}) + (-7)(1) + (0)(-2)}{\sqrt{(\sqrt{3})^2 + (-7)^2 + 0^2} \sqrt{(\sqrt{3})^2 + (1)^2 + (-2)^2}}\right) = \cos^{-1}\left(\frac{3-7}{\sqrt{52}\sqrt{8}}\right) = \cos^{-1}\left(\frac{-1}{\sqrt{26}}\right) \approx 1.77 \text{ rad}$$

$$2. \quad \theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\right) = \cos^{-1}\left(\frac{(1)(-1) + (\sqrt{2})(1) + (-\sqrt{2})(1)}{\sqrt{(1)^2 + (\sqrt{2})^2 + (-\sqrt{2})^2} \sqrt{(-1)^2 + (1)^2 + (1)^2}}\right) = \cos^{-1}\left(\frac{-1}{\sqrt{5}\sqrt{3}}\right) = \cos^{-1}\left(\frac{-1}{\sqrt{15}}\right) \approx 1.83 \text{ rad}$$

III. Find:

a. $\overrightarrow{PQ} \times \overrightarrow{PR}$

b. unit vector perpendicular to plane

1. $P(2, -2, 1)$, $Q(3, -1, 2)$, $R(3, -1, 1)$
2. $P(-2, 2, 0)$, $Q(0, 1, -1)$, $R(-1, 2, -2)$

Solution:

$$3. \quad (a) \quad \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -\mathbf{i} + \mathbf{j} \Rightarrow \text{Area } \frac{1}{2} |\overrightarrow{PQ} \times \overrightarrow{PR}| = \frac{1}{2} \sqrt{1+1} = \frac{\sqrt{2}}{2}$$

$$(b) \quad \mathbf{u} = \frac{\overrightarrow{PQ} \times \overrightarrow{PR}}{|\overrightarrow{PQ} \times \overrightarrow{PR}|} = \frac{1}{\sqrt{2}} (-\mathbf{i} + \mathbf{j}) \Rightarrow -\frac{1}{\sqrt{2}} (\mathbf{i} - \mathbf{j})$$

$$4. \quad (a) \quad \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & -1 \\ 1 & 0 & -2 \end{vmatrix} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k} \Rightarrow \text{Area} = \frac{1}{2} |\overrightarrow{PQ} \times \overrightarrow{PR}| = \frac{1}{2} \sqrt{4+9+1} = \frac{\sqrt{14}}{2}$$

$$(b) \quad \mathbf{u} = \frac{\overrightarrow{PQ} \times \overrightarrow{PR}}{|\overrightarrow{PQ} \times \overrightarrow{PR}|} = \frac{1}{\sqrt{14}} (2\mathbf{i} + 3\mathbf{j} + \mathbf{k})$$

iv. Verify that $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = (\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u} = (\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v}$

1. $2\mathbf{i} + \mathbf{j}$ $2\mathbf{i} - \mathbf{j} + \mathbf{k}$ $\mathbf{i} + 2\mathbf{k}$
2. $\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ $-\mathbf{i} - \mathbf{k}$ $2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$

Solution:

If $\mathbf{u} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{v} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, and $\mathbf{w} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$, then $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$$(\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u} = \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix} \text{ and } (\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v} = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$1. \quad |(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}| = \det \begin{vmatrix} 2 & 1 & 0 \\ 2 & -1 & 1 \\ 1 & 0 & 2 \end{vmatrix} = |-7| = 7$$

$$2. \quad |(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}| = \det \begin{vmatrix} 1 & 1 & -2 \\ -1 & 0 & -1 \end{vmatrix} = 8$$