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Mathematics (II)

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The Del and Curl Operators

The Del Operator

The del operator, denoted ∇, is an odd creature. It leads a double life as both differential operator and vector. In Cartesian coordinates on R^3, del is defined by:

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

$$\nabla f(x, y, z) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

The Divergence of a Vector Field

Let $F: X \subseteq R^n \to R^n$ be a differentiable vector field. Then the divergence of F, denoted div F or $\nabla \cdot F$ (the latter read "del dot F"), is the scalar field:

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \cdots + \frac{\partial F_n}{\partial x_n}$$

It turns a vector field into a scalar field:

Example:

$F = x^2 y\, i + xz\, j + xyz\, k$ Find $\nabla . F$

Sol:

$$\nabla . F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\nabla . F = 2xy + 0 + xy = 3xy$$

Example:

$F = x^2z^2\,i - 2y^2z^2\,j + xy^2z\,k$ Find $\nabla.F$ at the point P(1,-1, 1)

Sol:

$$\nabla.F = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}$$

$$\nabla.F = 2xz^2 - 4yz^2 + xy^2$$

$$\nabla.F = 2 + 4 + 1 = 7$$

The Curl of a Vector Field

Let F: X ⊆ R_3 → R_3 be a differentiable vector field on R3 only. The curl of F, denoted curl F or ∇×F (the latter read "del cross F"), is the vector field

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \times (F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k})$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}$$

It is still a vector field

Example:

$F = x^2yi - 2xzj + (x + y - z)k$ Find $\nabla \times F$

Sol:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2y & -2xz & x + y - z \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(x + y - z) - \frac{\partial}{\partial z}(-2xz)\right)\mathbf{i} + \left(\frac{\partial}{\partial z}(x^2y) - \frac{\partial}{\partial x}(x + y - z)\right)\mathbf{j}$$

$$+ \left(\frac{\partial}{\partial x}(-2xz) - \frac{\partial}{\partial y}(x^2y)\right)\mathbf{k}$$

$$= (1 + 2x)\mathbf{i} - \mathbf{j} - (x^2 + 2z)\mathbf{k}.$$

Example:

$A = x^2z^2i - 2y^2z^2j + xy^2zk$ Find $\nabla \times A$ at the point P(1,-1, 1)

Sol:

$$\nabla \times \mathbf{A} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \times (x^2z^2\mathbf{i} - 2y^2z^2\mathbf{j} + xy^2z\mathbf{k})$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2z^2 & -2y^2z^2 & xy^2z \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(xy^2z) - \frac{\partial}{\partial z}(-2y^2z^2)\right]\mathbf{i} - \left[\frac{\partial}{\partial x}(xy^2z) - \frac{\partial}{\partial z}(x^2z^2)\right]\mathbf{j} + \left[\frac{\partial}{\partial x}(-2y^2z^2) - \frac{\partial}{\partial y}(x^2z^2)\right]\mathbf{k}$$

$$= (2xyz + 4y^2z)\mathbf{i} - (y^2z - 2x^2z)\mathbf{j} + 0\mathbf{k}$$

At the point $P(1, -1, 1)$, $\nabla \times \mathbf{A} = 2\mathbf{i} + \mathbf{j}$

Partial Derivative of Vectors

Let F: $X \subseteq R^3 \rightarrow R^3$, the partial derivatives of F:

$$\frac{\partial^2 F}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial x}\right) \qquad \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right)$$

$$\frac{\partial^2 F}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial y}\right) \qquad \frac{\partial^2 F}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right)$$

$$\frac{\partial^2 F}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial z}\right) \qquad \frac{\partial^3 F}{\partial x \partial z^2} = \frac{\partial}{\partial x}\left(\frac{\partial^2 F}{\partial z^2}\right)$$

Rule(1): Suppose A and B are vector functions of x, y, z. Then the following laws hold

$$\text{(i)}\quad \frac{\partial}{\partial x}(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\cdot\frac{\partial \mathbf{B}}{\partial x} + \frac{\partial \mathbf{A}}{\partial x}\cdot\mathbf{B}$$

$$\text{(ii)}\quad \frac{\partial}{\partial x}(\mathbf{A}\times\mathbf{B}) = \mathbf{A}\times\frac{\partial \mathbf{B}}{\partial x} + \frac{\partial \mathbf{A}}{\partial x}\times\mathbf{B}$$

$$\text{(iii)}\quad \frac{\partial^2}{\partial y\partial x}(\mathbf{A}\cdot\mathbf{B}) = \frac{\partial}{\partial y}\left\{\frac{\partial}{\partial x}(\mathbf{A}\cdot\mathbf{B})\right\} = \frac{\partial}{\partial y}\left\{\mathbf{A}\cdot\frac{\partial \mathbf{B}}{\partial x} + \frac{\partial \mathbf{A}}{\partial x}\cdot\mathbf{B}\right\}$$

$$= \mathbf{A}\cdot\frac{\partial^2 \mathbf{B}}{\partial y\partial x} + \frac{\partial \mathbf{A}}{\partial y}\cdot\frac{\partial \mathbf{B}}{\partial x} + \frac{\partial \mathbf{A}}{\partial x}\cdot\frac{\partial \mathbf{B}}{\partial y} + \frac{\partial^2 \mathbf{A}}{\partial y\partial x}\cdot\mathbf{B}$$

Rule(2): Suppose A and B are vector functions of x, y, z. Then the following laws hold

(i) If $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$, then $d\mathbf{A} = dA_1\mathbf{i} + dA_2\mathbf{j} + dA_3\mathbf{k}$

(ii) $d(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\cdot d\mathbf{B} + d\mathbf{A}\cdot\mathbf{B}$

(iii) $d(\mathbf{A}\times\mathbf{B}) = \mathbf{A}\times d\mathbf{B} + d\mathbf{A}\times\mathbf{B}$

(iv) If $\mathbf{A} = \mathbf{A}(x, y, z)$, then $d\mathbf{A} = \frac{\partial \mathbf{A}}{\partial x}dx + \frac{\partial \mathbf{A}}{\partial y}dy + \frac{\partial \mathbf{A}}{\partial z}dz$, and so on

Example:

Suppose $\phi(x, y, z) = xy^2z$ and $\mathbf{A} = x\mathbf{i} + \mathbf{j} + xy\mathbf{k}$. Find $\frac{\partial^3}{\partial x^2\partial z}(\phi A)$ at the point $P(1, 2, 2)$.

Sol:

$$\phi A = x^2y^2z\mathbf{i} + xy^2z\mathbf{j} + x^2y^3z\mathbf{k}$$

$$\frac{\partial}{\partial z}(\phi A) = x^2y^2\mathbf{i} + xy^2\mathbf{j} + x^2y^3\mathbf{k}$$

$$\frac{\partial^2}{\partial x\partial z}(\phi A) = 2xy^2\mathbf{i} + y^2\mathbf{j} + 2xy^3\mathbf{k}$$

$$\frac{\partial^3}{\partial x^2\partial z}(\phi A) = 2y^2\mathbf{i} + 2y^3\mathbf{k}$$

When $x = 1$, $y = 2$, and $z = 2$, $\frac{\partial^3}{\partial x^2\partial z}(\phi A) = 8\mathbf{i} + 16\mathbf{k}$

Example:

A particle moves along a curve whose parametric equations are:

$$x = e^{-t},\ y = 2\cos 3t,\ z = 2\sin 3t$$

(a) Determine its velocity and acceleration at any time.
(b) Find the magnitudes of the velocity and acceleration at t = 0.

Sol:

(a) The position vector r of the particle is:

$$r = xi + yj + zk = e^{-t}\mathbf{i} + 2\cos 3t\mathbf{j} + 2\sin 3t\mathbf{k}$$

The velocity is: $v = d\mathbf{r}/dt = -e^{-t}\mathbf{i} - 6\sin 3t\mathbf{j} + 6\cos 3t\mathbf{k}$

The acceleration is: $\mathbf{a} = d^2\mathbf{r}/dt^2 = e^{-t}\mathbf{i} - 18\cos 3t\mathbf{j} - 18\sin 3t\mathbf{k}$

(b) magnitude of velocity at $t = 0$ is $\sqrt{(-1)^2 + (6)^2} = \sqrt{37}$

magnitude of acceleration at $t = 0$ is $\sqrt{(1)^2 + (-18)^2} = \sqrt{325}$

Example:

Given $\mathbf{R} = (3\cos t)\mathbf{i} + (3\sin t)\mathbf{j} + (4t)\mathbf{k}$. Find: (a) $\frac{d\mathbf{R}}{dt}$, (b) $\frac{d^2\mathbf{R}}{dt}$, (c) $\left|\frac{d\mathbf{R}}{dt}\right|$, (d) $\left|\frac{d^2\mathbf{R}}{dt}\right|$

Sol:

(a) $\frac{d\mathbf{R}}{dt} = \frac{d}{dt}(3\cos t)\mathbf{i} + \frac{d}{dt}(3\sin t)\mathbf{j} + \frac{d}{dt}(4t)\mathbf{k} = (-3\sin t)\mathbf{i} + (3\cos t)\mathbf{j} + 4\mathbf{k}.$

(b) $\frac{d^2\mathbf{R}}{dt} = \frac{d}{dt}\left(\frac{d\mathbf{R}}{dt}\right) = \frac{d}{dt}(-3\sin t)\mathbf{i} + \frac{d}{dt}(3\cos t)\mathbf{j} + \frac{d}{dt}(4)\mathbf{k} = (-3\cos t)\mathbf{i} + (-3\sin t)\mathbf{j}$

(c) $\left|\frac{d\mathbf{R}}{dt}\right| = [(-3\sin t)^2 + (3\cos t)^2 + (4)^2]^{1/2} = 5.$

(d) $\left|\frac{d^2\mathbf{R}}{dt}\right| = [(-3\cos t)^2 + (-3\sin t)^2 + (0)^2]^{1/2} = 3.$

Example:

Suppose A and B are differentiable functions of a scalar u. Prove

$$\frac{d}{du}(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\cdot\frac{d\mathbf{B}}{du} + \frac{d\mathbf{A}}{du}\cdot\mathbf{B}.$$

Sol:

Let $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$, $\mathbf{B} = B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}$. Then

$$\begin{aligned}\frac{d}{du}(\mathbf{A}\cdot\mathbf{B}) &= \frac{d}{du}(A_1B_1 + A_2B_2 + A_3B_3)\\ &= \left(A_1\frac{dB_1}{du} + A_2\frac{dB_2}{du} + A_3\frac{dB_3}{du}\right) + \left(\frac{dA_1}{du}B_1 + \frac{dA_2}{du}B_2 + \frac{dA_3}{du}B_3\right)\\ &= \mathbf{A}\cdot\frac{d\mathbf{B}}{du} + \frac{d\mathbf{A}}{du}\cdot\mathbf{B}\end{aligned}$$